

THE UNIVERSAL RELATION BETWEEN SCALING EXPONENTS IN FIRST-PASSAGE PERCOLATION

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ABSTRACT. It has been conjectured in numerous physics papers that in ordinary first-passage percolation on integer lattices, the fluctuation exponent χ and the wandering exponent ξ are related through the universal relation $\chi = 2\xi - 1$, irrespective of the dimension. This is sometimes called the KPZ relation between the two exponents. This article gives a rigorous proof of this conjecture assuming that the exponents exist in a certain sense.

1. INTRODUCTION

Consider the space \mathbb{R}^d with Euclidean norm $|\cdot|$, where $d \geq 2$. Consider \mathbb{Z}^d as a subset of this space, and say that two points x and y in \mathbb{Z}^d are nearest neighbors if $|x - y| = 1$. Let $E(\mathbb{Z}^d)$ be the set of nearest neighbor bonds in \mathbb{Z}^d . Let $t = (t_e)_{e \in E(\mathbb{Z}^d)}$ be a collection of i.i.d. non-negative random variables. In first-passage percolation, the variable t_e is usually called the ‘passage time’ through the edge e , alternately called the ‘edge-weight’ of e . We will sometimes refer to the collection t of edge-weights as the ‘environment’. The total passage time, or total weight, of a path P in the environment t is simply the sum of the weights of the edges in P and will be denoted by $t(P)$ in this article. The first-passage time $T(x, y)$ from a point x to a point y is the minimum total passage time among all lattice paths from x to y . For all our purposes, it will suffice to consider self-avoiding paths; henceforth, ‘lattice path’ will refer to only self-avoiding paths.

Note that if the edge-weights are continuous random variables, then with probability one there is a unique ‘geodesic’ between any two points x and y . This is denoted by $G(x, y)$ in this paper. Let $D(x, y)$ be the maximum deviation (in Euclidean distance) of this path from the straight line segment joining x and y (see Figure 1).

Although invented by mathematicians [11], the first-passage percolation and related models have attracted considerable attention in the theoretical physics literature (see [21] for a survey). Among other things, the physicists are particularly interested in two ‘scaling exponents’, sometimes denoted by χ and ξ in the mathematical physics literature. The *fluctuation exponent*

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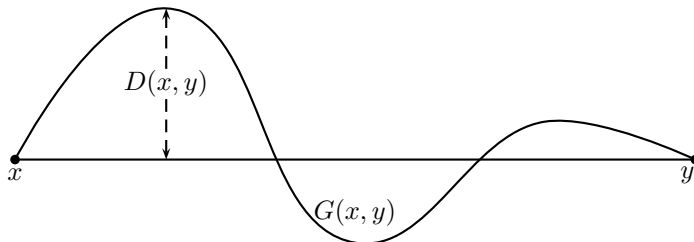


FIGURE 1. The geodesic $G(x, y)$ and the deviation $D(x, y)$.

χ is a number that quantifies the order of fluctuations of the first-passage time $T(x, y)$. Roughly speaking, for any x, y ,

the typical value of $T(x, y) - \mathbb{E}T(x, y)$ is of the order $|x - y|^\chi$.

The *wandering exponent* ξ quantifies the magnitude of $D(x, y)$. Again, roughly speaking, for any x, y ,

the typical value of $D(x, y)$ is of the order $|x - y|^\xi$.

There have been several attempts to give precise mathematical definitions for these exponents (see [23] for some examples) but I could not find a consensus in the literature. The main hurdle is that no one knows whether the exponents actually exist, and if they do, in what sense.

There are many conjectures related to χ and ξ . The main among these, to be found in numerous physics papers [14, 15, 16, 19, 20, 21, 24, 25, 30], including the famous paper of Kardar, Parisi and Zhang [15], is that although χ and ξ may depend on the dimension, they always satisfy the relation

$$\chi = 2\xi - 1.$$

A well-known conjecture from [15] is that when $d = 2$, $\chi = 1/3$ and $\xi = 2/3$. Yet another belief is that $\chi = 0$ if d is sufficiently large. Incidentally, due to its connection with [15], I've heard in private conversations the relation $\chi = 2\xi - 1$ being referred to as the 'KPZ relation' between χ and ξ .

There are a number of rigorous results for χ and ξ , mainly from the late eighties and early nineties. One of the first non-trivial results is due to Kesten [18, Theorem 1], who proved that $\chi \leq 1/2$ in any dimension. The only improvement on Kesten's result till date is due to Benjamini, Kalai and Schramm [6], who proved that for first-passage percolation in $d \geq 2$ with binary edge-weights,

$$(1) \quad \sup_{v \in \mathbb{Z}^d, |v| > 1} \frac{\text{Var} T(0, v)}{|v| / \log |v|} < \infty.$$

Benaïm and Rossignol [5] extended this result to a large class of edge-weight distributions that they call 'nearly gamma' distributions. The definition of a nearly gamma distribution is as follows. A positive random variable X is said to have a nearly gamma distribution if it has a continuous probability

density function h supported on an interval I (which may be unbounded), and its distribution function H satisfies, for all $y \in I$,

$$\Phi' \circ \Phi^{-1}(H(y)) \leq A\sqrt{y}h(y),$$

for some constant A , where Φ is the distribution function of the standard normal distribution. Although the definition may seem a bit strange, Benaïm and Rossignol [5] proved that this class is actually quite large, including e.g. exponential, gamma, beta and uniform distributions on intervals.

The only non-trivial lower bound on the fluctuations of passage times is due to Newman and Piza [26] and Pemantle and Peres [27], who showed that in $d = 2$, $\text{Var}T(0, v)$ must grow at least as fast as $\log |v|$. Better lower bounds can be proved if one can show that with high probability, the geodesics lie in ‘thin cylinders’ [7].

For the wandering exponent ξ , the main rigorous results are due to Licea, Newman and Piza [23] who showed that $\xi^{(2)} \geq 1/2$ in any dimension, and $\xi^{(3)} \geq 3/5$ when $d = 2$, where $\xi^{(2)}$ and $\xi^{(3)}$ are exponents defined in their paper which may be equal to ξ .

Besides the bounds on χ and ξ mentioned above, there are some rigorous results relating χ and ξ through inequalities. Wehr and Aizenman [29] proved the inequality $\chi \geq (1 - (d-1)\xi)/2$ in a related model, and the version of this inequality for first-passage percolation was proved by Licea, Newman and Piza [23]. The closest that anyone came to proving $\chi = 2\xi - 1$ is a result of Newman and Piza [26], who proved that $\chi' \geq 2\xi - 1$, where χ' is a related exponent which may be equal to χ . This has also been observed by Howard [13] under different assumptions.

Incidentally, in the model of Brownian motion in a Poissonian potential, Wüthrich [31] proved the equivalent of the KPZ relation assuming that the exponents exist.

The following theorem establishes the relation $\chi = 2\xi - 1$ assuming that the exponents χ and ξ exist in a certain sense (to be defined in the statement of the theorem) and that the distribution of edge-weights is nearly gamma.

Theorem 1.1. *Consider the first-passage percolation model on \mathbb{Z}^d , $d \geq 2$, with i.i.d. edge-weights. Assume that the distribution of edge-weights is ‘nearly gamma’ in the sense of Benaïm and Rossignol [5] (which includes exponential, gamma, beta and uniform distributions, among others), and has a finite moment generating function in a neighborhood of zero. Let χ_a and ξ_a be the smallest real numbers such that for all $\chi' > \chi_a$ and $\xi' > \xi_a$, there exists $\alpha > 0$ such that*

$$(A1) \quad \sup_{v \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E} \exp \left(\alpha \frac{|T(0, v) - \mathbb{E}T(0, v)|}{|v|^{\chi'}} \right) < \infty,$$

$$(A2) \quad \sup_{v \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E} \exp \left(\alpha \frac{D(0, v)}{|v|^{\xi'}} \right) < \infty.$$

Let χ_b and ξ_b be the largest real numbers such that for all $\chi' < \chi_b$ and $\xi' < \xi_b$, there exists $C > 0$ such that

$$(A3) \quad \inf_{v \in \mathbb{Z}^d, |v| > C} \frac{\text{Var}(T(0, v))}{|v|^{2\chi'}} > 0,$$

$$(A4) \quad \inf_{v \in \mathbb{Z}^d, |v| > C} \frac{\mathbb{E}D(0, v)}{|v|^{\xi'}} > 0.$$

Then $0 \leq \chi_b \leq \chi_a \leq 1/2$, $0 \leq \xi_b \leq \xi_a \leq 1$ and $\chi_a \geq 2\xi_b - 1$. Moreover, if it so happens that $\chi_a = \chi_b$ and $\xi_a = \xi_b$, and these two numbers are denoted by χ and ξ , then they must necessarily satisfy the relation $\chi = 2\xi - 1$.

Note that if $\chi_a = \chi_b$ and $\xi_a = \xi_b$ and these two numbers are denoted by χ and ξ , then χ and ξ are characterized by the properties that for every $\chi' > \chi$ and $\xi' > \xi$, there are some positive α and C such that for all $v \neq 0$,

$$\mathbb{E} \exp\left(\alpha \frac{|T(0, v) - \mathbb{E}T(0, v)|}{|v|^{\chi'}}\right) < C \quad \text{and} \quad \mathbb{E} \exp\left(\alpha \frac{D(0, v)}{|v|^{\xi'}}\right) < C,$$

and for every $\chi' < \chi$ and $\xi' < \xi$ there are some positive B and C such that for all v with $|v| > C$,

$$\text{Var}(T(0, v)) > B|v|^{2\chi'} \quad \text{and} \quad \mathbb{E}D(0, v) > B|v|^{\xi'}.$$

It seems reasonable to expect that if the two exponents χ and ξ indeed exist, then they should satisfy the above properties.

Incidentally, a few months after the first draft of this paper was put up on arXiv, Auffinger and Damron [4] were able to replace a crucial part of the proof of Theorem 1.1 with a simpler argument that allowed them to remove the assumption that the edge-weights are nearly-gamma.

Section 2 has a sketch of the proof of Theorem 1.1. The rest of the paper is devoted to the actual proof. Proving that $0 \leq \chi_b \leq \chi_a \leq 1/2$ and $0 \leq \xi_b \leq \xi_a \leq 1$ is a routine exercise; this is done in Section 3. Proving that $\chi_a \geq 2\xi_b - 1$ is also relatively easy and similar to the existing proofs of analogous inequalities, e.g. in [26, 13]. This is done in Section 6. The ‘hard part’ is proving the opposite inequality, that is, $\chi \leq 2\xi - 1$ when $\chi = \chi_a = \chi_b$ and $\xi = \xi_a = \xi_b$. This is done in Sections 7, 8 and 9.

2. PROOF SKETCH

I will try to give a sketch of the proof in this section. I have found it very hard to aptly summarize the main ideas in the proof without going into the details. This proof-sketch represents the end-result of my best efforts in this direction. If the interested reader finds the proof sketch too obscure, I would like to request him to return to this section after going through the complete proof, whereupon this high-level sketch may shed some illuminating insights.

Throughout this proof sketch, C will denote any positive constant that depends only on the edge-weight distribution and the dimension. Let $h(x) :=$

$\mathbb{E}(T(0, x))$. The function h is subadditive. Therefore the limit

$$g(x) := \lim_{n \rightarrow \infty} \frac{h(nx)}{n}$$

exists for all $x \in \mathbb{Z}^d$. The definition can be extended to all $x \in \mathbb{Q}^d$ by taking $n \rightarrow \infty$ through a subsequence, and can be further extended to all $x \in \mathbb{R}^d$ by uniform continuity. The function g is a norm on \mathbb{R}^d .

The function g is a norm, and hence much more well-behaved than h . If $|x|$ is large, $g(x)$ is supposed to be a good approximation of $h(x)$. A method developed by Ken Alexander [1, 2] uses the order of fluctuations of passage times to infer bounds on $|h(x) - g(x)|$. In the setting of Theorem 1.1, Alexander's method yields that for any $\varepsilon > 0$, there exists C such that for all $x \neq 0$,

$$(2) \quad g(x) \leq h(x) \leq g(x) + C|x|^{\chi_a + \varepsilon}.$$

This is formally recorded in Theorem 4.1. In the proof of the main result, the above approximation will allow us to replace the expected passage time $h(x)$ by the norm $g(x)$.

In Lemma 5.1, we prove that there is a unit vector x_0 and a hyperplane H_0 perpendicular to x_0 such that for some $C > 0$, for all $z \in H_0$,

$$|g(x_0 + z) - g(x_0)| \leq C|z|^2.$$

Similarly, there is a unit vector x_1 and a hyperplane H_1 perpendicular to x_1 such that for some $C > 0$, for all $z \in H_1$, $|z| \leq 1$,

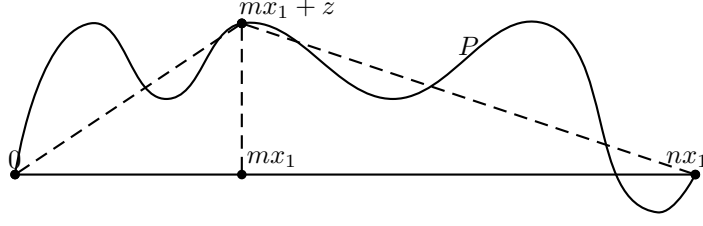
$$g(x_1 + z) \geq g(x_1) + C|z|^2.$$

The interpretations of these two inequalities is as follows. In the direction x_0 , the unit sphere of the norm g is 'at most as curved as an Euclidean sphere' and in the direction x_1 , it is 'at least as curved as an Euclidean sphere'.

Now take a look at Figure 2. Think of m as a fraction of n . By the definition of the direction of curvature x_1 and Alexander's approximation (2), for any $\varepsilon > 0$,

$$\begin{aligned} & \text{Expected passage time of the path } P \\ & \geq g(mx_1 + z) + g(nx_1 - (mx_1 + z)) + O(n^{\chi + \varepsilon}) \\ & = mg(x_1 + z/m) + (n - m)g(x_1 + z/(n - m)) + O(n^{\chi + \varepsilon}) \\ & \geq ng(x_1) + C|z|^2/n + O(n^{\chi + \varepsilon}) \\ & \geq \mathbb{E}(T(0, nx_1)) + C|z|^2/n + O(n^{\chi + \varepsilon}). \end{aligned}$$

Suppose $|z| = n^\xi$. Then $|z|^2/n = n^{2\xi - 1}$. Fluctuations of $T(0, nx_1)$ are of order n^χ . Thus, if $2\xi - 1 > \chi$, then P cannot be a geodesic from 0 to nx_1 . This sketch is formalized into a rigorous argument in Section 6 to prove that $\chi_a \geq 2\xi_b - 1$.

FIGURE 2. Proving $\chi \geq 2\xi - 1$

Next, let me sketch the proof of $\chi \leq 2\xi - 1$ when $\chi > 0$. The methods developed in [7] for first-passage percolation in thin cylinders have some bearing on this part of the proof. Recall the direction of curvature x_0 . Let $a = n^\beta$, $\beta < 1$. Let $m = n/a = n^{1-\beta}$. Under the conditions $\chi > 2\xi - 1$ and $\chi > 0$, we will show that there is a $\beta < 1$ such that

$$(\star) \quad T(0, nx_0) = \sum_{i=0}^{m-1} T(iax_0, (i+1)ax_0) + o(n^\chi).$$

This will lead to a contradiction, as follows. Let $f(n) := \text{Var}T(0, nx_0)$. Then by Benaïm and Rossignol [5], $f(n) \leq Cn/\log n$. Under (\star) , by the Harris-FKG inequality,

$$\begin{aligned} f(n) &= \text{Var}T(0, nx_0) \geq m \text{Var}T(0, ax_0) + o(n^{2\chi}) \\ &= n^{1-\beta} f(n^\beta) + o(n^{2\chi}). \end{aligned}$$

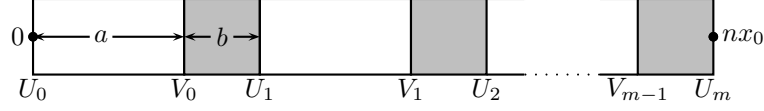
If β is chosen sufficiently small, the first term on the right will dominate the second. Consequently,

$$(\dagger) \quad \liminf_{n \rightarrow \infty} \frac{f(n)}{n^{1-\beta} f(n^\beta)} \geq 1.$$

Choose $n_0 > 1$ and define $n_{i+1} = n_i^{1/\beta}$ for each i . Let $v(n) := f(n)/n$. Then $v(n_i) \leq C/\log n_i \leq C\beta^i$. But by (\dagger) , $\liminf v(n_{i+1})/v(n_i) \geq 1$, and so for all i large enough, $v(n_{i+1}) \geq \beta^{1/2} v(n_i)$. In particular, there is a positive constant c such that for all i , $v(n_i) \geq c\beta^{i/2}$. Since $\beta < 1$, this gives a contradiction for i large, therefore proving that $\chi \leq 2\xi - 1$.

Let me now sketch a proof of (\star) under the conditions $\chi > 2\xi - 1$ and $\chi > 0$. Let $a = n^\beta$ and $b = n^{\beta'}$, where $\beta' < \beta < 1$. Consider a cylinder of width n^ξ around the line joining 0 and nx_0 . Partition the cylinder into alternating big and small cylinders of widths a and b respectively. Call the boundary walls of these cylinders $U_0, V_0, U_1, V_1, \dots, V_{m-1}, U_m$, where m is roughly $n^{1-\beta}$ (see Figure 3).

Let $G_i := G(U_i, V_i)$, that is, the path with minimum passage time between any vertex in U_i and any vertex in V_i . Let u_i and v_i be the endpoints of G_i .

FIGURE 3. Cylinder of width n^ξ around the line joining 0 and nx_0

Let $G'_i := G(v_i, u_{i+1})$. The concatenation of the paths $G'_0, G_1, G'_1, G_2, \dots, G'_{m-1}, G_m$ is a path from U_0 to U_m . Therefore,

$$T(U_0, U_m) \leq \sum_{i=1}^{m-1} T(U_i, V_i) + \sum_{i=0}^{m-1} T(v_i, u_{i+1}).$$

Next, let $G := G(U_0, U_m)$. Let u'_i be the first vertex in U_i visited by G and let v'_i be the first vertex in V_i visited by G . If G stays within the cylinder throughout, then $T(u'_i, v'_i) \geq T(U_i, V_i)$ and $T(v'_i, u'_{i+1}) \geq T(V_i, U_{i+1})$. Thus,

$$T(U_0, U_m) \geq \sum_{i=0}^{m-1} T(U_i, V_i) + \sum_{i=0}^{m-1} T(V_i, U_{i+1}).$$

Thus, if $G(U_0, U_m)$ stays in a cylinder of width n^ξ , then

$$\begin{aligned} 0 &\leq T(U_0, U_m) - \sum_{i=0}^{m-1} (T(U_i, V_i) + T(V_i, U_{i+1})) \\ &\leq \sum_{i=0}^{m-1} (T(v_i, u_{i+1}) - T(V_i, U_{i+1})). \end{aligned}$$

Therefore,

$$\left| T(U_0, U_m) - \sum_{i=0}^{m-1} (T(U_i, V_i) + T(V_i, U_{i+1})) \right| \leq \sum_{i=0}^{m-1} M_i,$$

where $M_i := \max_{v, v' \in V_i, u, u' \in U_{i+1}} |T(v, u) - T(v', u')|$. Note that the errors M_i come only from the small blocks. By curvature estimate in direction x_0 , for any $v, v' \in V_i$ and $u, u' \in U_{i+1}$,

$$|\mathbb{E}T(v, u) - \mathbb{E}T(v', u')| \leq C(n^\xi)^2/n^{\beta'} = Cn^{2\xi-\beta'}.$$

Fluctuations of $T(v, u)$ are of order $n^{\beta'\chi}$. If $2\xi - 1 < \chi$, then we can choose β' so close to 1 that $2\xi - \beta' < \beta'\chi$. That is, fluctuations dominate while estimating M_i . Consequently, M_i is of order $n^{\beta'\chi}$. Thus, total error = $n^{1-\beta+\beta'\chi}$. Since $\beta' < \beta$ and $\chi > 0$, this gives us the opportunity of choosing β', β such that the exponent is $< \chi$. This proves (\star) for passage times from

‘boundary to boundary’. Proving (\star) for ‘point to point’ passage times is only slightly more complicated. The program is carried out in Sections 7 and 8.

Finally, for the case $\chi = 0$, we have to prove that $\xi \geq 1/2$. This was proved by Licea, Newman and Piza [23] for a different definition of the wandering exponent. The argument does not seem to work with our definition. A proof is given in Section 9; I will omit this part from the proof sketch.

3. A PRIORI BOUNDS

In this section we prove the a priori bounds $0 \leq \chi_b \leq \chi_a \leq 1/2$ and $0 \leq \xi_b \leq \xi_a \leq 1$. First, note that the inequalities $\chi_b \leq \chi_a$ and $\xi_b \leq \xi_a$ are easy. For example, if $\chi_b > \chi_a$, then for any $\chi_a < \chi' < \chi'' < \chi_b$, (A1) implies that

$$\sup_{v \in \mathbb{Z}^d \setminus \{0\}} \frac{\text{Var}(T(0, v))}{|v|^{2\chi'}} < \infty,$$

and hence for any sequence v_n such that $|v_n| \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \frac{\text{Var}(T(0, v_n))}{|v_n|^{2\chi''}} = 0,$$

which contradicts (A3). A similar argument shows that $\xi_b \leq \xi_a$.

To show that $\chi_b \geq 0$, let E_0 denote the set of all edges incident to the origin. Let \mathcal{F}_0 denote the sigma-algebra generated by $(t_e)_{e \notin E_0}$. Since the edge-weight distribution is non-degenerate, there exists $c_1 < c_2$ such that for an edge e , $\mathbb{P}(t_e < c_1) > 0$ and $\mathbb{P}(t_e > c_2) > 0$. Therefore,

$$(3) \quad \mathbb{P}(\max_{e \in E_0} t_e < c_1) > 0, \quad \mathbb{P}(\min_{e \in E_0} t_e > c_2) > 0.$$

Let $(t'_e)_{e \in E_0}$ be an independent configuration of edge weights. Define $t'_e = t_e$ if $e \notin E_0$. Let $T'(0, v)$ be the first-passage time from 0 to a vertex v in the new environment t' . If $t_e < c_1$ and $t'_e > c_2$ for all $e \in E_0$, then $T'(0, v) > T(0, v) + c_2 - c_1$. Thus, by (3), there exists $\delta > 0$ such that for any v with $|v| \geq 2$,

$$\mathbb{E}\text{Var}(T(0, v) | \mathcal{F}_0) = \frac{1}{2} \mathbb{E}(T(0, v) - T'(0, v))^2 > \delta.$$

Therefore $\text{Var}(T(0, v)) > \delta$ and so $\chi_b \geq 0$.

To show that $\xi_b \geq 0$, note that there is an $\epsilon > 0$ small enough such that for any $v \in \mathbb{Z}^d$ with $|v| \geq 2$, there can be at most one lattice path from 0 to v that stays within distance ϵ from the straight line segment joining 0 to v . Fix such a vertex v and such a path P . If the number of edges in P is sufficiently large, one can use the non-degeneracy of the edge-weight distribution to show by an explicit assignment of edge weights that

$$\mathbb{P}(P \text{ is a geodesic}) < \delta,$$

where $\delta < 1$ is a constant that depends only on the edge-weight distribution (and not on v or P). This shows that for $|v|$ sufficiently large, $\mathbb{E}D(0, v)$ is

bounded below by a positive constant that does not depend on v , thereby proving that $\xi_b \geq 0$.

Let us next show that $\chi_a \leq 1/2$. Essentially, this follows from [18, Theorem 1] or [28, Proposition 8.3], with a little bit of extra work. Below, we give a proof using [5, Theorem 5.4]. First, note that there is a constant C_0 such that for all v ,

$$(4) \quad \mathbb{E}T(0, v) \leq C_0|v|_1,$$

where $|v|_1$ is the ℓ_1 norm of v . From the assumptions about the distribution of edge-weights, [5, Theorem 5.4] implies that there are positive constants C_1 and C_2 such that for any $v \in \mathbb{Z}^d$ with $|v|_1 \geq 2$, and any $0 \leq t \leq |v|_1$,

$$(5) \quad \mathbb{P}\left(|T(0, v) - \mathbb{E}T(0, v)| \geq t\sqrt{\frac{|v|_1}{\log |v|_1}}\right) \leq C_1 e^{-C_2 t}.$$

Fix a path P from 0 to v with $|v|_1$ edges. Recall that $t(P)$ denotes the sum of the weights of the edges in P . Since the edge-weight distribution has finite moment generating function in a neighborhood of zero and (4) holds, it is easy to see that there are positive constants C_3 , C_4 and C'_4 such that if $|v|_1 > C_3$, then for any $t > |v|_1$,

$$\begin{aligned} (6) \quad & \mathbb{P}\left(|T(0, v) - \mathbb{E}T(0, v)| \geq t\sqrt{\frac{|v|_1}{\log |v|_1}}\right) \\ & \leq \mathbb{P}\left(T(0, v) \geq C_0|v|_1 + t\sqrt{\frac{|v|_1}{\log |v|_1}}\right) \\ & \leq \mathbb{P}\left(t(P) \geq C_0|v|_1 + t\sqrt{\frac{|v|_1}{\log |v|_1}}\right) \leq e^{C_4|v|_1 - C'_4 t\sqrt{|v|_1/\log |v|_1}}. \end{aligned}$$

Combining (5) and (6) it follows that there are constants C_5 , C_6 and C_7 such that for any v with $|v|_1 > C_5$,

$$\mathbb{E} \exp\left(C_6 \frac{|T(0, v) - \mathbb{E}T(0, v)|}{\sqrt{|v|_1/\log |v|_1}}\right) \leq C_7.$$

Appropriately increasing C_7 , one sees that the above inequality holds for all v with $|v|_1 \geq 2$. In particular, $\chi_a \leq 1/2$.

Finally, let us prove that $\xi_a \leq 1$. Consider a self-avoiding path P starting at the origin, containing m edges. By the strict positivity of the edge-weight distributions, for any edge e ,

$$\lim_{\theta \rightarrow \infty} \mathbb{E}(e^{-\theta t_e}) = 0.$$

Now, for any $\theta, c > 0$,

$$\mathbb{P}(t(P) \leq cm) = \mathbb{P}(e^{-t(P)/c} \geq e^{-m}) \leq (e\mathbb{E}(e^{-t_e/c}))^m.$$

Thus, given any $\delta > 0$ there exists c small enough such that for any m and any self-avoiding path P with m edges,

$$\mathbb{P}(t(P) \leq cm) \leq \delta^m.$$

Since there are at most $(2d)^m$ paths with m edges, therefore there exists c small enough such that

$$\mathbb{P}(t(P) \leq cm \text{ for some } P \text{ with } m \text{ edges}) \leq 2^{-m-1},$$

and therefore

$$(7) \quad \mathbb{P}(t(P) \leq cm \text{ for some } P \text{ with } \geq m \text{ edges}) \leq 2^{-m}.$$

There is a constant $B > 0$ such that for any $t \geq 1$ and any vertex $v \neq 0$, if $D(0, v) \geq t|v|$, then $G(0, v)$ has at least $Bt|v|$ edges. Therefore from (7),

$$\mathbb{P}(D(0, v) \geq t|v|) \leq \mathbb{P}(T(0, v) \geq Bt|v|/c) + 2^{-Bt|v|}.$$

As in (6), there is a constant C such that if P is a path from 0 to v with $|v|_1$ edges,

$$\mathbb{P}(T(0, v) \geq Bt|v|/c) \leq \mathbb{P}(t(P) \geq Bt|v|/c) \leq e^{C|v| - Bt|v|/c}.$$

Combining the last two displays shows that for some α small enough,

$$\sup_{v \neq 0} \mathbb{E} \exp\left(\alpha \frac{D(0, v)}{|v|}\right) < \infty,$$

and thus, $\xi_a \leq 1$.

4. ALEXANDER'S SUBADDITIVE APPROXIMATION THEORY

The first step in the proof of Theorem 1.1 is to find a suitable approximation of $\mathbb{E}T(0, x)$ by a convex function $g(x)$. For $x \in \mathbb{Z}^d$, define

$$(8) \quad h(x) := \mathbb{E}T(0, x).$$

It is easy to see that h satisfies the subadditive inequality

$$h(x + y) \leq h(x) + h(y).$$

By the standard subadditive argument, it follows that

$$(9) \quad g(x) := \lim_{n \rightarrow \infty} \frac{h(nx)}{n}$$

exists for each $x \in \mathbb{Z}^d$. In fact, $g(x)$ may be defined similarly for $x \in \mathbb{Q}^d$ by taking $n \rightarrow \infty$ through a sequence of n such that $nx \in \mathbb{Z}^d$. The function g extends continuously to the whole of \mathbb{R}^d , and the extension is a norm on \mathbb{R}^d (see e.g. [2, Lemma 1.5]). Note that by subadditivity,

$$(10) \quad g(x) \leq h(x) \text{ for all } x \in \mathbb{Z}^d.$$

Since the edge-weight distribution is continuous in the setting of Theorem 1.1, it follows by a well-known result (see [17]) that $g(x) > 0$ for each

$x \neq 0$. Let e_i denote the i th coordinate vector in \mathbb{R}^d . Since g is symmetric with respect to interchange of coordinates and reflections across all coordinate hyperplanes, it is easy to show using subadditivity that

$$(11) \quad |x|_\infty \leq g(x)/g(e_1) \leq |x|_1 \text{ for all } x \neq 0,$$

where $|x|_p$ denotes the ℓ_p norm of the vector x .

How well does $g(x)$ approximate $h(x)$? Following the work of Kesten [17, 18], Alexander [1, 2] developed a general theory for tackling such questions. One of the main results of Alexander [2] is that under appropriate hypotheses on the edge-weights, there exists some $C > 0$ such that for all $x \in \mathbb{Z}^d \setminus \{0\}$,

$$g(x) \leq h(x) \leq g(x) + C|x|^{1/2} \log |x|.$$

Incidentally, Alexander has recently been able to obtain slightly improved results for nearly gamma edge-weights [3]. It turns out that under the hypotheses of Theorem 1.1, Alexander's argument goes through almost verbatim to yield the following result.

Theorem 4.1. *Consider the setup of Theorem 1.1. Let g and h be defined as in (9) and (8) above. Then for any $\chi' > \chi_a$, there exists $C > 0$ such that for all $x \in \mathbb{Z}^d$ with $|x| > 1$,*

$$g(x) \leq h(x) \leq g(x) + C|x|^{\chi'} \log |x|.$$

Sacrificing brevity for the sake of completeness, I will now prove Theorem 4.1 by copying Alexander's argument with only minor changes at the appropriate points.

Fix $\chi' > \chi_a$. Since $0 \leq \chi_a \leq 1/2$, so χ' can be chosen to satisfy $0 < \chi' < 1$.

Let $B_0 := \{x : g(x) \leq 1\}$. Given $x \in \mathbb{R}^d$, let H_x denote a hyperplane tangent to the boundary of $g(x)B_0$ at x . Note that if the boundary is not smooth, the choice of H_x may not be unique. Let H_x^0 be the hyperplane through the origin that is parallel to H_x . There is a unique linear functional g_x on \mathbb{R}^d satisfying

$$g_x(y) = 0 \text{ for all } y \in H_x^0, \quad g_x(x) = g(x).$$

For each $x \in \mathbb{R}^d$, $C > 0$ and $K > 0$ let

$$Q_x(C, K)$$

$$:= \{y \in \mathbb{Z}^d : |y| \leq K|x|, \quad g_x(y) \leq g(x), \quad h(y) \leq g_x(y) + C|x|^{\chi'} \log |x|\}.$$

The following key result is taken from [2].

Lemma 4.2 (Alexander [2], Theorem 1.8). *Consider the setting of Theorem 4.1. Suppose that for some $M > 1$, $C > 0$, $K > 0$ and $a > 1$, the following holds. For each $x \in \mathbb{Q}^d$ with $|x| \geq M$, there exists an integer $n \geq 1$, a lattice path γ from 0 to nx , and a sequence of sites $0 = v_0, v_1, \dots, v_m = nx$ in γ such that $m \leq an$ and $v_i - v_{i-1} \in Q_x(C, K)$ for all $1 \leq i \leq m$. Then the conclusion of Theorem 4.1 holds.*

Before proving that the conditions of Lemma 4.2 hold, we need some preliminary definitions and results. Define

$$s_x(y) := h(y) - g_x(y), \quad y \in \mathbb{Z}^d.$$

By the definition of g_x and the fact that g is a norm, it is easy to see that

$$(12) \quad |g_x(y)| \leq g(y),$$

and by subadditivity, $g(y) \leq h(y)$. Therefore $s_x(y) \geq 0$. Again from subadditivity of h and linearity of g_x ,

$$(13) \quad s_x(y+z) \leq s_x(y) + s_x(z) \quad \text{for all } y, z \in \mathbb{Z}^d.$$

Let $C_1 := 320d^2/\alpha$, where α is from the statement of Theorem 1.1. As in [2], define

$$\begin{aligned} Q_x &:= Q_x(C_1, 2d+1), \\ G_x &:= \{y \in \mathbb{Z}^d : g_x(y) > g(x)\}, \\ \Delta_x &:= \{y \in Q_x : y \text{ adjacent to } \mathbb{Z}^d \setminus Q_x, \text{ } y \text{ not adjacent to } G_x\}, \\ D_x &:= \{y \in Q_x : y \text{ adjacent to } G_x\}. \end{aligned}$$

The following Lemma is simply a slightly altered copy of Lemma 3.3 in [2].

Lemma 4.3. *Assume the conditions of Theorem 1.1. Then there exists a constant C_2 such that if $|x| \geq C_2$, the following hold.*

- (i) *If $y \in Q_x$ then $g(y) \leq 2g(x)$ and $|y| \leq 2d|x|$.*
- (ii) *If $y \in \Delta_x$ then $s_x(y) \geq C_1|x|^{\chi'}(\log|x|)/2$.*
- (iii) *If $y \in D_x$ then $g_x(y) \geq 5g(x)/6$.*

Proof. (i) Suppose $g(y) > 2g(x)$ and $g_x(y) \leq g(x)$. Then using (10) and (12),

$$2g(x) < g(y) \leq h(y) = g_x(y) + s_x(y) \leq g(x) + s_x(y),$$

so from (11), $s_x(y) > g(x) > C_1|x|^{\chi'} \log|x|$ provided $|x| \geq C_2$. Thus $y \notin Q_x$ and the first conclusion in (i) follows. The second conclusion then follows from (11).

(ii) Note that $z = y \pm e_i$ for some $z \in \mathbb{Z}^d \cap Q_x^c \cap G_x^c$ and $i \leq d$. From (i) we have $|y| \leq 2d|x|$, so $|z| \leq (2d+1)|x|$, provided $|x| > 1$. Since $z \notin Q_x$ we must then have $s_x(z) > C_1|x|^{\chi'} \log|x|$, while using (12),

$$h(\pm e_i) = s_x(\pm e_i) + g_x(\pm e_i) \geq s_x(\pm e_i) - g(\pm e_i).$$

Consequently, by (13), if $|x| \geq C_2$,

$$\begin{aligned} s_x(y) &\geq s_x(z) - s_x(\pm e_i) \\ &\geq C_1|x|^{\chi'} \log|x| - h(\pm e_i) - g(\pm e_i) \\ &\geq C_1|x|^{\chi'} (\log|x|)/2. \end{aligned}$$

(iii) As in (ii) we have $z = y \pm e_i$ for some $z \in \mathbb{Z}^d \cap G_x$ and $i \leq d$. Therefore using (11) and (12),

$$g_x(y) = g_x(z) - g_x(\pm e_i) \geq g_x(z) - g(\pm e_i) \geq 5g(x)/6$$

for all $|x| \geq C_2$. \square

Let us call the $m+1$ sites in Lemma 4.2 marked sites. If m is unrestricted, it is easy to find inductively a sequence of marked sites for any path γ from 0 to nx , as follows. One can start at $v_0 = 0$, and given v_i , let v'_{i+1} be the first site (if any) in γ , coming after v_i , such that $v'_{i+1} - v_i \notin Q_x$; then let v_{i+1} be the last site in γ before v'_{i+1} if v'_{i+1} exists; otherwise let $v_{i+1} = nx$ and end the construction. If $|x|$ is large enough, then it is easy to deduce from (11) and (12) that all neighbors of the origin must belong to Q_x and therefore $v_{i+1} \neq v_i$ for each i and hence the construction must end after a finite number of steps. We call the sequence of marked sites obtained from a self-avoiding path γ in this way, the Q_x -skeleton of γ .

Given such a skeleton (v_0, \dots, v_m) , abbreviated (v_i) , of some lattice path, we divide the corresponding indices into two classes, corresponding to ‘long’ and ‘short’ increments:

$$\begin{aligned} S((v_i)) &:= \{i : 0 \leq i < m-1, v_{i+1} - v_i \in \Delta_x\}, \\ L((v_i)) &:= \{i : 0 \leq i < m-1, v_{i+1} - v_i \in D_x\}. \end{aligned}$$

Note that the final index m is in neither class, and by Lemma 4.3(ii),

$$(14) \quad j \in S((v_i)) \text{ implies } s_x(v_{j+1} - v_j) > C_1 |x|^{\chi'} (\log |x|)/2.$$

The next result is analogous to Proposition 3.4 in [2].

Proposition 4.4. *Assume the conditions of Theorem 1.1. There exists a constant C_3 such that if $|x| \geq C_3$ then for sufficiently large n there exists a lattice path from 0 to nx with Q_x -skeleton of $2n+1$ or fewer vertices.*

Proof. Let (v_0, \dots, v_m) be a Q_x -skeleton of some lattice path and let

$$Y_i := \mathbb{E}T(v_i, v_{i+1}) - T(v_i, v_{i+1}).$$

Then by (A1) of Theorem 1.1 and Lemma 4.3(i), there are constants $C_4 := \alpha/(2d)^{\chi'} \geq \alpha/2d$ and C_5 such that for $0 \leq i \leq m-1$,

$$(15) \quad \mathbb{E} \exp(C_4 |Y_i| / |x|^{\chi'}) \leq C_5.$$

Let $Y'_0, Y'_1, \dots, Y'_{m-1}$ be independent random variables with Y'_i having the same distribution as Y_i . Let $T(0, w; (v_j))$ be the minimum passage time among all lattice paths from 0 to a site w with Q_x -skeleton (v_j) . By [17, equation (4.13)] or [1, Theorem 2.3], for all $t \geq 0$,

$$\mathbb{P}\left(\sum_{i=0}^{m-1} Y'_i \geq t\right) \geq \mathbb{P}\left(\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \geq t\right).$$

Now by (15),

$$\mathbb{P}\left(\sum_{i=0}^{m-1} Y'_i \geq t\right) \leq e^{-C_4 t/|x|^{\chi'}} C_5^m.$$

Let $C_6 := 20d^2/\alpha$. Taking $t = C_6 m |x|^{\chi'} \log |x|$, the above display shows that there is a constant C_7 such that for all $|x| \geq C_7$,

$$\mathbb{P}\left(\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \geq C_6 m |x|^{\chi'} \log |x|\right) \leq (C_5 e^{-10d \log |x|})^m.$$

From the definition of a Q_x -skeleton, it is easy to see that there is a constant C_8 such that there are at most $(C_8 |x|^d)^m$ Q_x -skeletons with $m+1$ vertices. Therefore, the above display shows that there are constants C_9 and C_{10} such that when $|x| \geq C_9$,

$$\mathbb{P}\left(\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \geq C_6 m |x|^{\chi'} \log |x| \text{ for some } Q_x\text{-skeleton with } m+1 \text{ vertices}\right) \leq e^{-C_{10} m \log |x|}.$$

This in turn yields that for some constant C_{11} , for all $|x| \geq C_{11}$,

$$\begin{aligned} & \mathbb{P}\left(\sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, v_m; (v_j)) \geq C_6 m |x|^{\chi'} \log |x| \right. \\ (16) \quad & \left. \text{for some } m \geq 1 \text{ and some } Q_x\text{-skeleton with } m+1 \text{ vertices}\right) \\ & \leq 2e^{-C_{10} \log |x|}. \end{aligned}$$

Now let $\omega := \{t_e : e \text{ is an edge in } \mathbb{Z}^d\}$ be a fixed configuration of passage times (to be further specified later) and let (v_0, \dots, v_m) be the Q_x -skeleton of a route from 0 to nx . Then since $v_{i+1} - v_i \in Q_x$,

$$mg(x) \geq \sum_{i=0}^{m-1} g_x(v_{i+1} - v_i) = g_x(nx) = ng(x).$$

Therefore

$$(17) \quad n \leq m.$$

From the concentration of first-passage times,

$$\mathbb{P}(T(0, nx) \leq ng(x) + n) \rightarrow 1 \text{ as } n \rightarrow \infty,$$

so by (16) if n is large there exists a configuration ω and a Q_x -skeleton (v_0, \dots, v_m) of a path from 0 to nx such that

$$(18) \quad T(0, nx; (v_j)) = T(0, nx) \leq ng(x) + n$$

and

$$(19) \quad \sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) - T(0, nx; (v_j)) < C_6 m |x|^{\chi'} \log |x|.$$

Thus for some constant C_{12} , if $|x| \geq C_{12}$ then by (17), (18) and (19),

$$(20) \quad \begin{aligned} \sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) &< ng(x) + n + C_6 m |x|^{\chi'} \log |x| \\ &\leq ng(x) + 2C_6 m |x|^{\chi'} \log |x|. \end{aligned}$$

But by (14),

$$\begin{aligned} \sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) &= \sum_{i=0}^{m-1} (g_x(v_{i+1} - v_i) + s_x(v_{i+1} - v_i)) \\ &\geq g_x(nx) + C_1 |S((v_i))| |x|^{\chi'} (\log |x|)/2, \end{aligned}$$

which, together with (20), yields

$$(21) \quad |S((v_i))| \leq 4C_6 m / C_1 = m/4.$$

At the same time, using Lemma 4.3(iii),

$$\begin{aligned} \sum_{i=0}^{m-1} \mathbb{E}T(v_i, v_{i+1}) &= \sum_{i=0}^{m-1} (g_x(v_{i+1} - v_i) + s_x(v_{i+1} - v_i)) \\ &\geq 5|L((v_i))|g(x)/6. \end{aligned}$$

With (20), (11) and the assumption that $\chi' < 1$, this implies that there is a constant C_{13} such that, provided $|x| \geq C_{13}$,

$$|L((v_i))| \leq 6n/5 + \frac{12C_6 m |x|^{\chi'} \log |x|}{6g(e_1)|x|/\sqrt{d}} \leq 6n/5 + m/8.$$

This and (21) give

$$m = |L((v_i))| + |S((v_i))| + 1 \leq 6n/5 + 3m/8 + 1,$$

which, for n large, implies $m \leq 2n$, proving the Proposition. \square

Proof of Theorem 4.1. Lemma 4.2 and Proposition 4.4 prove the conclusion of Theorem 4.1 for x with sufficiently large Euclidean norm. To prove this for all x with $|x| > 1$, one simply has to increase the value of C . \square

5. CURVATURE BOUNDS

The unit ball of the g -norm, usually called the ‘limit shape’ of first-passage percolation, is an object of great interest and intrigue in this literature. Very little is known rigorously about the limit shape, except for a fundamental result about convergence to the limit shape due to Cox and Durrett [8], some qualitative results of Kesten [17] who proved, in particular, that the limit shape may not be an Euclidean ball, an important result of Durrett and

Liggett [9] who showed that the boundary of the limit shape may contain straight lines, and some bounds on the rate of convergence to the limit shape [18, 2]. In particular, it is not even known whether the limit shape may be strictly convex in every direction (except for the related continuum model of ‘Riemannian first-passage percolation’ [22] and first-passage percolation with stationary ergodic edge-weights [10]).

The following Proposition lists two properties of the limit shape that are crucial for our purposes.

Proposition 5.1. *Let g be defined as in (9) and assume that the distribution of edge-weights is continuous. Then there exists $x_0 \in \mathbb{R}^d$ with $|x_0| = 1$, a constant $C \geq 0$ and a hyperplane H_0 through the origin perpendicular to x_0 such that for all $z \in H_0$,*

$$|g(x_0 + z) - g(x_0)| \leq C|z|^2.$$

There also exists $x_1 \in \mathbb{R}^d$ with $|x_1| = 1$ and a hyperplane H_1 through the origin perpendicular to x_1 such that for all $z \in H_1$,

$$g(x_1 + z) \geq \sqrt{1 + |z|^2}g(x_1).$$

Proof. The proof is similar to that of [26, Lemma 5]. Let $B(0, r)$ denote the Euclidean ball of radius r centered at the origin and let

$$B_g(0, r) := \{x : g(x) \leq r\}$$

denote the ball of radius r centered at the origin for the norm g . Let r be the smallest number such that $B_g(0, r) \supseteq B(0, 1)$. Let x_0 be a point of intersection of $\partial B_g(0, r)$ and $\partial B(0, 1)$. Let H_0 be a hyperplane tangent to $\partial B_g(0, r)$ at x_0 , translated to contain the origin. Note that $x_0 + H_0$ is also a tangent hyperplane for $B(0, 1)$ at x_0 , since it touches $B(0, 1)$ only at x_0 . Therefore H_0 is perpendicular to x_0 . Now for any $z \in H_0$, the point $y := (x_0 + z)/|x_0 + z|$ is a point on $\partial B(0, 1)$ and hence contained in $B_g(0, r)$. Therefore

$$g(x_0) = r \geq g(y) = \frac{1}{|x_0 + z|}g(x_0 + z) = \frac{1}{\sqrt{1 + |z|^2}}g(x_0 + z).$$

Since $g(x_0 + z)$ grows like $|z|$ as $|z| \rightarrow \infty$, this shows that there is a constant C such that

$$g(x_0 + z) \leq g(x_0) + C|z|^2$$

for all $z \in H_0$. Also, since $x_0 + z \notin B_g(0, r)$ for $z \in H_0 \setminus \{0\}$, therefore $g(x_0) \leq g(x_0 + z)$ for all $z \in H_0$. This proves the first assertion of the Proposition.

For the second, we proceed similarly. Let r be the largest number such that $B_g(0, r) \subseteq B(0, 1)$. Let x_1 be a point in the intersection of $\partial B_g(0, r)$ and $\partial B(0, 1)$. Let H_1 be the hyperplane tangent to $\partial B(0, 1)$ at x_1 , translated to contain the origin. Note that this is simply the hyperplane through

the origin that is perpendicular to x_1 . Since $B(0, 1)$ contains $B_g(0, r)$, and $y := (x_1 + z)/|x_1 + z|$ is a point in $\partial B(0, 1)$, therefore

$$g(x_1) = r \leq g(y) = \frac{1}{|x_1 + z|} g(x_1 + z) = \frac{1}{\sqrt{1 + |z|^2}} g(x_1 + z).$$

This completes the argument. \square

6. PROOF OF $\chi_a \geq 2\xi_b - 1$

We will prove by contradiction. Suppose that $2\xi_b - 1 > \chi_a$. Choose ξ' such that

$$\frac{1 + \chi_a}{2} < \xi' < \xi_b.$$

Note that $\xi' < 1$. Let x_1 and H_1 be as in Proposition 5.1. Let n be a positive integer, to be chosen later. Throughout this proof, C will denote any positive constant that does not depend on n . The value of C may change from line to line. Also, we will assume without mention that ‘ n is large enough’ wherever required.

Let y be the closest point in \mathbb{Z}^d to nx_1 . Note that

$$(22) \quad |y - nx_1| \leq \sqrt{d}.$$

Let L denote the line passing through 0 and nx_1 and let L' denote the line segment joining 0 to nx_1 (but not including the endpoints). Let V be the set of all points in \mathbb{Z}^d whose distance from L' lies in the interval $[n^{\xi'}, 2n^{\xi'}]$. Take any $v \in V$. We claim that there is a constant C (not depending on n) such that for any $v \in V$,

$$(23) \quad g(v) + g(nx_1 - v) \geq g(nx_1) + Cn^{2\xi' - 1}.$$

Let us now prove this claim. Let w be the projection of v onto L along H_1 (i.e. the perpendicular projection). To prove (23), there are three cases to consider. First suppose that w lies in L' . Note that $w/|w| = x_1$. Let $v' := v/|w|$ and $z := v' - x_1 = (v - w)/|w|$.

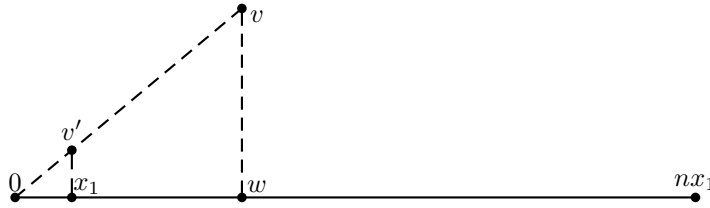


FIGURE 4. The relative positions of x_1, v', v, w, nx_1 .

Note that $z \in H_1$. Thus by Proposition 5.1,

$$g(v') = g(x_1 + z) \geq \sqrt{1 + |z|^2} g(x_1).$$

Consequently,

$$(24) \quad g(v) \geq |w| \sqrt{1 + |z|^2} g(x_1).$$

Next, let $w' := nx_1 - w$. Note that $w'/|w'| = x_1$. let $v'' := (nx_1 - v)/|w'|$, and

$$z' := v'' - x_1 = (w - v)/|w'|.$$

Then $z' \in H_1$, and hence by Proposition 5.1,

$$g(v'') = g(x_1 + z') \geq \sqrt{1 + |z'|^2} g(x_1).$$

Consequently,

$$(25) \quad g(nx_1 - v) \geq |w'| \sqrt{1 + |z'|^2} g(x_1).$$

Since $v \in V$, therefore $|v - w| \geq n^{\xi'}$. Again, $|w| + |w'| = n$. Thus,

$$\min\{|z|, |z'|\} \geq n^{\xi'-1}.$$

Combining this with (24), (25), (11) and the fact that $\xi' < 1$, we have

$$\begin{aligned} g(v) + g(nx_1 - v) &\geq (|w| + |w'|) \sqrt{1 + n^{2\xi'-2}} g(x_1) \\ &= \sqrt{1 + n^{2\xi'-2}} g(nx_1) \\ &\geq g(nx_1) + Cn^{2\xi'-1}. \end{aligned}$$

Next, suppose that w lies in $L \setminus L'$, on the side closer to nx_1 . As above, let $z := (v - w)/|w|$. As in (24), we conclude that

$$(26) \quad g(v) \geq |w| \sqrt{1 + |z|^2} g(x_1).$$

By the definition of V , the distance between v and nx_1 must be greater than $n^{\xi'}$. But in this case

$$|v - nx_1|^2 = (|w| - n)^2 + |v - w|^2 = (|w| - n)^2 + |w|^2 |z|^2,$$

and we also have $n \leq |w| \leq 3n$. Thus, either $|w|^2 |z|^2 > n^{2\xi'}/2$ (which implies $|z|^2 \geq Cn^{2\xi'-2}$), or $|w| \geq n + n^{\xi'}/\sqrt{2}$. Since $\xi' > 2\xi' - 1$, therefore by (26), in either situation we have

$$g(v) \geq g(nx_1) + Cn^{2\xi'-1}.$$

Similarly, if w lies in $L \setminus L'$, on the side closer to 0, then

$$g(nx_1 - v) \geq g(nx_1) + Cn^{2\xi'-1}.$$

This completes the proof of (23). Now (23) combined with Theorem 4.1, (22) and the fact that $2\xi' - 1 > \chi_a$ implies that if n is large enough, then for any $v \in V$,

$$(27) \quad h(v) + h(y - v) \geq h(y) + Cn^{2\xi'-1}.$$

Choose χ_1, χ_2 such that $\chi_a < \chi_1 < \chi_2 < 2\xi' - 1$. Then by (A1) of Theorem 1.1, there is a constant C such that for n large enough,

$$\mathbb{P}(T(0, y) > h(y) + n^{\chi_2}) \leq e^{-Cn^{\chi_2 - \chi_1}}.$$

Now, for any $v \in V$, both $|v|$ and $|y-v|$ are bounded above by Cn . Therefore again by (A1),

$$\begin{aligned}\mathbb{P}(T(0, v) < h(v) - n^{\chi_2}) &\leq e^{-Cn^{\chi_2 - \chi_1}}, \\ \mathbb{P}(T(v, y) < h(y - v) - n^{\chi_2}) &\leq e^{-Cn^{\chi_2 - \chi_1}}.\end{aligned}$$

This, together with (27), shows that if n is large enough, then for any $v \in V$,

$$\mathbb{P}(T(0, y) = T(0, v) + T(v, y)) \leq e^{-Cn^{\chi_2 - \chi_1}}.$$

Since the size of V grows polynomially with n , this shows that

$$\mathbb{P}(T(0, y) = T(0, v) + T(v, y) \text{ for some } v \in V) \leq e^{-Cn^{\chi_2 - \chi_1}}.$$

Note that if the geodesic from 0 to y passes through V , then $T(0, y) = T(0, v) + T(v, y)$ for some $v \in V$. If $D(0, y) > n^{\xi'}$ then the geodesic must pass through V . Thus, the above inequality implies that

$$\mathbb{P}(D(0, y) > n^{\xi'}) \leq e^{-Cn^{\chi_2 - \chi_1}}.$$

By (A2) of Theorem 1.1, this gives

$$\begin{aligned}\mathbb{E}D(0, y) &\leq n^{\xi'} + \mathbb{E}(D(0, y)1_{\{D(0, y) > n^{\xi'}\}}) \\ &\leq n^{\xi'} + \sqrt{\mathbb{E}(D(0, y)^2)\mathbb{P}(D(0, y) > n^{\xi'})} \\ &\leq n^{\xi'} + C_1 n^{C_1} e^{-C_2 n^{\chi_2 - \chi_1}}.\end{aligned}$$

Taking $n \rightarrow \infty$, this shows that (A4) of Theorem 1.1 is violated (since $\xi' < \xi_b$), leading to a contradiction to our original assumption that $\chi_a < 2\xi_b - 1$. Thus, $\chi_a \geq 2\xi_b - 1$.

7. PROOF OF $\chi \leq 2\xi - 1$ WHEN $0 < \chi < 1/2$

In this section and the rest of the manuscript, we assume that $\chi_a = \chi_b$ and $\xi_a = \xi_b$, and denote these two numbers by χ and ξ .

Again we prove by contradiction. Suppose that $0 < \chi < 1/2$ and $\chi > 2\xi - 1$. Fix $\chi_1 < \chi < \chi_2$, to be chosen later. Choose ξ' such that

$$\xi < \xi' < \frac{1 + \chi}{2}.$$

Define:

$$\begin{aligned}\beta' &:= \frac{1}{2} + \frac{\xi'}{1 + \chi}, \\ \beta &:= 1 - \frac{\chi}{2} + \frac{\chi}{2}\beta', \\ \varepsilon &:= (1 - \beta)\left(1 - \frac{\chi}{2}\right).\end{aligned}$$

We need several inequalities involving the numbers β' , β and ε . Since

$$0 < \frac{\xi'}{1 + \chi} < \frac{1}{2},$$

therefore

$$(28) \quad \frac{1}{2} < \beta' < 1.$$

Since $\chi < 1$ and $\xi' < (1 + \chi)/2 < 1$,

$$(29) \quad \beta' > \frac{1}{2} + \frac{\xi'}{2} > \xi'.$$

Since β is a convex combination of 1 and β' and $\chi > 0$,

$$(30) \quad \beta' < \beta < 1.$$

Since $0 < \chi < 1$ and $0 < \beta < 1$,

$$(31) \quad 0 < \varepsilon < 1 - \beta.$$

Since β' is the average of 1 and $2\xi'/(1 + \chi) \in (0, 1)$, therefore β' is strictly bigger than $2\xi'/(1 + \chi)$ and hence

$$(32) \quad \begin{aligned} 2\xi' - \beta' &< 2\xi' - \frac{2\xi'}{1 + \chi} \\ &= \frac{2\xi'}{1 + \chi}\chi < \beta'\chi. \end{aligned}$$

By (30), this implies that

$$(33) \quad 2\xi' - \beta < 2\xi' - \beta' < \beta'\chi < \beta\chi.$$

Next, by (28),

$$(34) \quad 1 - \beta + \beta'\chi = \frac{\chi}{2}(1 + \beta') < \chi.$$

And finally by (28),

$$(35) \quad \beta\chi + 1 - \beta - \varepsilon = \beta\chi + (1 - \beta)\frac{\chi}{2} < \chi.$$

Let q be a large positive integer, to be chosen later. Throughout this proof, we will assume without mention that q is ‘large enough’ wherever required. Also, C will denote any constant that does not depend on our choice of q , but may depend on all other parameters.

Let r be an integer between $\frac{1}{2}q^{(1-\beta-\varepsilon)/\varepsilon}$ and $2q^{(1-\beta-\varepsilon)/\varepsilon}$, recalling that by (31), $1 - \beta - \varepsilon > 0$. Let $k = rq$. Let a be a real number between $q^{\beta/\varepsilon}$ and $2q^{\beta/\varepsilon}$. Let $n = ak$. Note that $n = arq$, which gives $\frac{1}{2}q^{1/\varepsilon} \leq n \leq 4q^{1/\varepsilon}$. From this it is easy to see that there are positive constants C_1 and C_2 , depending only on β and ε , such that

$$(36) \quad C_1 n^\varepsilon \leq q \leq C_2 n^\varepsilon,$$

$$(37) \quad C_1 n^{1-\beta} \leq k \leq C_2 n^{1-\beta},$$

$$(38) \quad C_1 n^\beta \leq a \leq C_2 n^\beta,$$

$$(39) \quad C_1 n^{1-\beta-\varepsilon} < r < C_2 n^{1-\beta-\varepsilon}.$$

Let $b := n^{\beta'}$. Note that by (30), b is negligible compared to a if q is large. Note also that, although r, k and q are integers, a, n and b need not be.

Let x_0 and H_0 be as in Proposition 5.1. For $0 \leq i \leq k$, define

$$\begin{aligned} U'_i &:= H_0 + iax_0, \\ V'_i &:= H_0 + (ia + a - b)x_0. \end{aligned}$$

Let U_i be the set of points in \mathbb{Z}^d that are within distance \sqrt{d} from U'_i . Let V_i be the set of points in \mathbb{Z}^d that are within distance \sqrt{d} from V'_i .

For $0 \leq i \leq k$ let y_i be the closest point in \mathbb{Z}^d to iax_0 , and let z_i be the closest point in \mathbb{Z}^d to $(ia + a - b)x_0$, applying some arbitrary rule to break ties. Note that if $x \in \mathbb{R}^d$, and $y \in \mathbb{Z}^d$ is closest to x , then $|x - y| \leq \sqrt{d}$. Therefore $y_i \in U_i$ and $z_i \in V_i$. Figure 5 gives a pictorial representation of the above definitions, assuming for simplicity that $U_i = U'_i$ and $V_i = V'_i$.

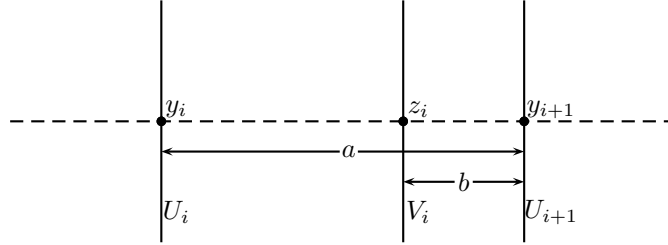


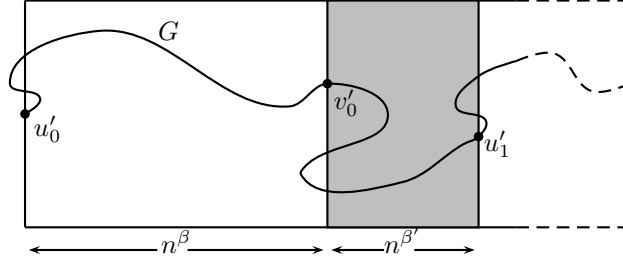
FIGURE 5. Diagrammatic representation of y_i, z_i, U_i and V_i .

Let U_i^o be the subset of U_i that is within distance $n^{\xi'}$ from y_i . Similarly let V_i^o be the subset of V_i that is within distance $n^{\xi'}$ from z_i .

For any $A, B \subseteq \mathbb{Z}^d$, let $T(A, B)$ denote the minimum passage time from A to B . Let $G(A, B)$ denote the (unique) geodesic from A to B , so that $T(A, B)$ is the sum of edge-weights of $G(A, B)$.

Fix any two integers $0 \leq l < m \leq k$ such that $m - l > 3$. Consider the geodesic $G := G(y_l, y_m)$. Since $x_0 \notin H_0$, it is easy to see that G must ‘hit’ each U_i and V_i , $l \leq i \leq m - 1$. Arranging the vertices of G in a sequence starting at y_l and ending at y_m , for each $l \leq i < m$ let u'_i be the first vertex in U_i visited by G and let v'_i be the first vertex in V_i visited by G . Let $u'_m := y_m$. Note that G visits these vertices in the order $u'_l, v'_l, u'_{l+1}, v'_{l+1}, \dots, v'_{m-1}, u'_m$. Figure 6 gives a pictorial representation of the points u'_i and v'_i on the geodesic G . Let T'_i be the sum of edge-weights of the portion of G from u'_i to v'_i . Let E be the event that $u'_i \in U_i^o$ and $v'_i \in V_i^o$ for each i . If E happens, then clearly

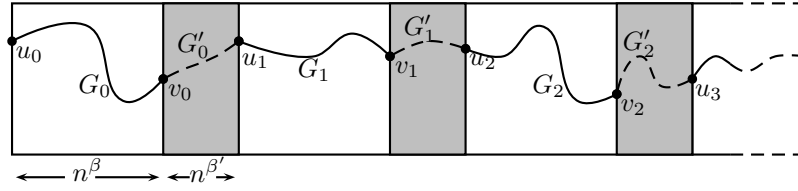
$$T'_i \geq T(U_i^o, V_i^o).$$

FIGURE 6. Location of $u'_0, v'_0, u'_1, v'_1, \dots$ on the geodesic G .

Similarly, note that weight of the part of G from v'_i to u'_{i+1} must exceed or equal $T(v'_i, u'_{i+1})$. Therefore, if E happens, then

$$\begin{aligned}
 (40) \quad T(y_l, y_m) &\geq \sum_{i=l}^{m-1} T'_i + \sum_{i=l}^{m-1} T(v'_i, u'_{i+1}) \\
 &\geq \sum_{i=l}^{m-1} T(U_i^o, V_i^o) + \sum_{i=l}^{m-1} T(v'_i, u'_{i+1}).
 \end{aligned}$$

Next, for each $0 \leq i < k$ let $G_i := G(U_i^o, V_i^o)$. Let u_i and v_i be the endpoints of G_i . Let $G'_i := G(v_i, u_{i+1})$. Figure 7 gives a picture illustrating the paths G_i and G'_i . The concatenation of the paths $G(y_l, v_l), G'_l, G_{l+1}, G'_{l+1}, G_{l+2},$

FIGURE 7. The paths $G_0, G'_0, G_1, G'_1, \dots$

$\dots, G'_{m-2}, G_{m-1}, G(v_{m-1}, y_m)$ is a path from y_l to y_m (need not be self-avoiding). Therefore,

$$\begin{aligned}
 (41) \quad T(y_l, y_m) &\leq T(y_l, v_l) + \sum_{i=l+1}^{m-1} T(U_i^o, V_i^o) + \sum_{i=l}^{m-2} T(v_i, u_{i+1}) \\
 &\quad + T(v_{m-1}, y_m).
 \end{aligned}$$

Define

$$\Delta_{l,m} := T(y_l, y_m) - \sum_{i=l}^{m-1} (T(U_i^o, V_i^o) + T(V_i^o, U_{i+1}^o)).$$

Combining (40) and (41) implies that if E happens, then

$$\begin{aligned} |\Delta_{l,m}| &\leq \sum_{i=l}^{m-1} |T(V_i^o, U_{i+1}^o) - T(v'_i, u'_{i+1})| + \sum_{i=l}^{m-2} |T(V_i^o, U_{i+1}^o) - T(v_i, u_{i+1})| \\ &\quad + |T(U_l^o, V_l^o) - T(y_l, v_l)| + |T(V_{m-1}^o, U_m^o) - T(v_{m-1}, y_m)|. \end{aligned}$$

Thus, if

$$\begin{aligned} M_i &:= \max_{v, v' \in V_i^o, u, u' \in U_{i+1}^o} |T(v, u) - T(v', u')|, \\ N_i &:= \max_{u, u' \in U_i^o, v, v' \in V_i^o} |T(u, v) - T(u', v')|, \end{aligned}$$

and the event E happens, then

$$(42) \quad |\Delta_{l,m}| \leq 2 \sum_{i=l}^{m-1} M_i + N_l.$$

For a random variable X , let $\|X\|_p := (\mathbb{E}|X|^p)^{1/p}$ denote its L^p norm. It is easy to see that $\|\Delta_{l,m}\|_4 \leq n^C$, where recall that C stands for any constant that does not depend on our choice of the integer q , but may depend on χ , ξ , ξ' and the distribution of edge weights. Take any $\xi_1 \in (\xi, \xi')$. By (A2) of Theorem 1.1, $\mathbb{P}(E^c) \leq e^{-Cn^{\xi'} - \xi_1}$. Together with (42), this shows that for some constants C_3 and C_4 ,

$$\begin{aligned} \|\Delta_{l,m}\|_2 &\leq \|\Delta_{l,m} 1_{E^c}\|_2 + \|\Delta_{l,m} 1_E\|_2 \\ &\leq \|\Delta_{l,m}\|_4 (\mathbb{P}(E^c))^{1/4} + \|\Delta_{l,m} 1_E\|_2 \\ (43) \quad &\leq n^{C_3} e^{-C_4 n^{\xi'} - \xi_1} + 2 \sum_{i=l}^{m-1} \|M_i\|_2 + \|N_l\|_2. \end{aligned}$$

Fix $0 \leq i \leq k-1$ and $v \in V_i^o$, $u \in U_{i+1}^o$. Let x be the nearest point to v in V'_i and y be the nearest point to u in U'_{i+1} . Then by definition of V'_i and U'_{i+1} , there are vectors $z, z' \in H_0$ such that $|z|$ and $|z'|$ are bounded by $Cn^{\xi'}$, and $x = (ia + a - b)x_0 + z$ and $y = (ia + a)x_0 + z'$. Thus by Proposition 5.1,

$$\begin{aligned} |g(y - x) - g(bx_0)| &= |g(bx_0 + z' - z) - g(bx_0)| \\ &= b|g(x_0 + (z' - z)/b) - g(x_0)| \\ &\leq \frac{C|z' - z|^2}{b} \leq Cn^{2\xi' - \beta'}. \end{aligned}$$

Thus, for any $v, v' \in V_i^o$ and $u, u' \in U_{i+1}^o$,

$$|g(u - v) - g(u' - v')| \leq Cn^{2\xi' - \beta'}.$$

Note also that $|y - x| \leq C(n^{\beta'} + n^{\xi'}) \leq Cn^{\beta'}$ by (29). This, together with Theorem 4.1, shows that for any $v, v' \in V_i^o$ and $u, u' \in U_{i+1}^o$,

$$|\mathbb{E}T(v, u) - \mathbb{E}T(v', u')| \leq Cn^{2\xi' - \beta'} + Cn^{\beta'\chi_2} \log n.$$

By (32), this implies

$$(44) \quad |\mathbb{E}T(v, u) - \mathbb{E}T(v', u')| \leq Cn^{\beta'\chi_2} \log n.$$

Let

$$M := \max_{v \in V_i^o, u \in U_{i+1}^o} \frac{|T(v, u) - \mathbb{E}T(v, u)|}{|u - v|^{\chi_2}}.$$

By (A1) of Theorem 1.1,

$$\begin{aligned} \mathbb{E}(e^{\alpha M}) &\leq \sum_{v \in V_i^o, u \in U_{i+1}^o} \mathbb{E} \exp \left(\alpha \frac{|T(v, u) - \mathbb{E}T(v, u)|}{|u - v|^{\chi_2}} \right) \\ &\leq C|V_i^o||U_{i+1}^o| \leq Cn^C. \end{aligned}$$

This implies that $\mathbb{P}(M > t) \leq Cn^C e^{-\alpha t}$, which in turn gives $\|M\|_2 \leq C \log n$.

Let

$$M' := \max_{v \in V_i^o, u \in U_{i+1}^o} |T(v, u) - \mathbb{E}T(v, u)|.$$

Since by (29), $|u - v| \leq C(n^{\beta'} + n^{\xi'}) \leq Cn^{\beta'}$ for all $v \in V_i^o, u \in U_{i+1}^o$, therefore $M' \leq Cn^{\beta'\chi_2} M$. Thus,

$$\|M'\|_2 \leq Cn^{\beta'\chi_2} \log n.$$

From this and (44) it follows that

$$\|M_i\|_2 \leq Cn^{\beta'\chi_2} \log n.$$

By an exactly similar sequence of steps, replacing β' by β everywhere and using (33) instead of (32), one can deduce that

$$\|N_i\|_2 \leq Cn^{\beta\chi_2} \log n.$$

Combining with (43) this gives

$$(45) \quad \|\Delta_{l,m}\|_2 \leq Cn^{\beta\chi_2} \log n + C(m-l)n^{\beta'\chi_2} \log n,$$

since the exponential term in (43) is negligible compared to the rest.

Now, from the definition of $\Delta_{l,m}$, the fact that $k = rq$, and the triangle inequality, it is easy to see that

$$\left| T(y_0, y_k) - \sum_{j=0}^{r-1} T(y_{jq}, y_{(j+1)q}) \right| \leq |\Delta_{0,k}| + \sum_{j=0}^{r-1} |\Delta_{jq, (j+1)q}|.$$

Thus by (45), (39) and (37),

$$\begin{aligned}
 (46) \quad \left\| T(y_0, y_k) - \sum_{j=0}^{r-1} T(y_{jq}, y_{(j+1)q}) \right\|_2 &\leq \|\Delta_{0,k}\|_2 + \sum_{j=0}^{r-1} \|\Delta_{jq, (j+1)q}\|_2 \\
 &\leq C(r+1)n^{\beta\chi_2} \log n + Ckn^{\beta'\chi_2} \log n \\
 &\leq Cn^{1-\beta-\varepsilon+\beta\chi_2} \log n + Cn^{1-\beta+\beta'\chi_2} \log n.
 \end{aligned}$$

For any two random variables X and Y ,

$$\begin{aligned}
 (47) \quad |\sqrt{\text{Var}(X)} - \sqrt{\text{Var}(Y)}| &= |\|X - \mathbb{E}X\|_2 - \|Y - \mathbb{E}Y\|_2| \\
 &\leq \|(X - \mathbb{E}X) - (Y - \mathbb{E}Y)\|_2 \\
 &\leq \|X - Y\|_2 + |\mathbb{E}X - \mathbb{E}Y| \leq 2\|X - Y\|_2.
 \end{aligned}$$

Therefore it follows from (46) that

$$\begin{aligned}
 (48) \quad \left| (\text{Var}T(y_0, y_k))^{1/2} - \left(\text{Var} \sum_{j=0}^{r-1} T(y_{jq}, y_{(j+1)q}) \right)^{1/2} \right| \\
 \leq Cn^{1-\beta-\varepsilon+\beta\chi_2} \log n + Cn^{1-\beta+\beta'\chi_2} \log n.
 \end{aligned}$$

For any $x, y \in \mathbb{Z}^d$, $T(x, y)$ is an increasing function of the edge weights. So by the Harris-FKG inequality [12], $\text{Cov}(T(x, y), T(x', y')) \geq 0$ for any $x, y, x', y' \in \mathbb{Z}^d$. Therefore by (A3) of Theorem 1.1 and (38), (39) and (36),

$$\begin{aligned}
 (49) \quad \text{Var} \sum_{j=0}^{r-1} T(y_{jq}, y_{(j+1)q}) &\geq \sum_{j=0}^{r-1} \text{Var}T(y_{jq}, y_{(j+1)q}) \\
 &\geq C \sum_{j=0}^{r-1} |y_{jq} - y_{(j+1)q}|^{2\chi_1} \\
 &\geq Cr(aq)^{2\chi_1} \geq Cn^{(1-\beta-\varepsilon)+(\beta+\varepsilon)2\chi_1}.
 \end{aligned}$$

By the inequalities (34) and (35), we see that if χ_1 and χ_2 are chosen sufficiently close to χ , then χ_1 is strictly bigger than both $1 - \beta - \varepsilon + \beta\chi_2$ and $1 - \beta + \beta'\chi_2$. Therefore by (48) and (49), and since $1 - \beta - \varepsilon + (\beta + \varepsilon)2\chi_1 > 2\chi_1$,

$$\text{Var}T(y_0, y_k) \geq Cn^{(1-\beta-\varepsilon)+(\beta+\varepsilon)2\chi_1}.$$

By (31) and the assumption that $\chi < 1/2$, we again have that if χ_1 is chosen sufficiently close to χ ,

$$(1 - \beta - \varepsilon) + (\beta + \varepsilon)2\chi_1 > 2\chi.$$

Since $|y_0 - y_k| \leq Cak \leq Cn$ by (38) and (37), therefore taking $q \rightarrow \infty$ (and hence $n \rightarrow \infty$) gives a contradiction to (A1) of Theorem 1.1, thereby proving that $\chi \leq 2\xi - 1$ when $0 < \chi < 1/2$.

8. PROOF OF $\chi \leq 2\xi - 1$ WHEN $\chi = 1/2$

Suppose that $\chi = 1/2$ and $\chi > 2\xi - 1$. Define $\chi_1, \chi_2, x_0, H_0, \xi', \beta, \beta', \varepsilon, q, a, r, k, n, y_i$ and z_i exactly as in Section 7, considering a, r, k and n as functions of q . Then all steps go through, except the very last, where we used $\chi < 1/2$ to get a contradiction. Therefore all we need to do is the modify this last step to get a contradiction in a different way. This is where we need the sublinear variance inequality (1). As before, throughout the proof C denotes any constant that does not depend on q .

For each real number $m \geq 1$, let w_m be the nearest lattice point to mx_0 . Note that $y_i = w_{ia}$. Let

$$f(m) := \text{Var}T(0, w_m).$$

Note that there is a constant C_0 such that $f(m) \leq C_0 m$ for all m . Again by (A3), there is a $C_1 > 0$ such that for all m ,

$$(50) \quad f(m) \geq C_1 m^{2\chi_1}.$$

Now, $|(w_{(j+1)aq} - w_{jaq}) - w_{aq}| \leq C$. Again, as a consequence of (47) we have that for any two random variables X and Y ,

$$(51) \quad \begin{aligned} |\text{Var}(X) - \text{Var}(Y)| &= |\sqrt{\text{Var}(X)} - \sqrt{\text{Var}(Y)}|(\sqrt{\text{Var}(X)} + \sqrt{\text{Var}(Y)}) \\ &\leq 2\|X - Y\|_2(2\sqrt{\text{Var}(X)} + 2\|X - Y\|_2). \end{aligned}$$

By (51) and the subadditivity of first-passage times,

$$\begin{aligned} \text{Var}(T(w_{jaq}, w_{(j+1)aq})) &\geq f(aq) - C\sqrt{f(aq)} - C \\ &\geq f(n/r) - C\sqrt{n/r}. \end{aligned}$$

Therefore by the Harris-FKG inequality,

$$(52) \quad \text{Var}\left(\sum_{j=0}^{r-1} T(w_{jaq}, w_{(j+1)aq})\right) \geq rf(n/r) - C\sqrt{nr}.$$

Now, by (34) and (35), if χ_2 is sufficiently close to χ , then both $1 - \beta - \varepsilon + \beta\chi_2$ and $1 - \beta + \beta'\chi_2$ are strictly smaller than $1/2$. Therefore by (46), (51) and the fact that $f(n) \leq Cn$,

$$\begin{aligned} &\left| f(n) - \text{Var}\left(\sum_{j=0}^{r-1} T(w_{jaq}, w_{(j+1)aq})\right) \right| \\ &\leq C\sqrt{n}(n^{1-\beta-\varepsilon+\beta\chi_2} \log n + n^{1-\beta+\beta'\chi_2} \log n). \end{aligned}$$

Combining this with (52) gives

$$f(n) \geq rf(n/r) - C\sqrt{nr} - C\sqrt{n}(n^{1-\beta-\varepsilon+\beta\chi_2} \log n + n^{1-\beta+\beta'\chi_2} \log n).$$

Again by (39) and (50),

$$rf(n/r) \geq Cn^{(1-\beta-\varepsilon)+(\beta+\varepsilon)2\chi_1}.$$

Combining (39) with the last two displays, it follows that we can choose χ_1 and χ_2 so close to $1/2$ that as $q \rightarrow \infty$,

$$\liminf \frac{f(n)}{rf(n/r)} \geq 1.$$

In particular, for any $\delta > 0$, there exists an integer $q(\delta)$ such that if $q \geq q(\delta)$, then

$$(53) \quad f(n) \geq (1 - \delta)rf(n/r).$$

Fix $\delta = (1 - \beta - \varepsilon)/2$ and choose $q(\delta)$ satisfying the above criterion. Note that $q(\delta)$ can be chosen as large as we like. Let $m_0 := aq = n/r$ and $m_1 = n$. The above inequality implies that

$$\frac{f(m_1)}{m_1} \geq (1 - \delta) \frac{f(m_0)}{m_0}.$$

Note that by (36), if $q(\delta)$ is chosen sufficiently large to begin with, then

$$m_1^{\varepsilon/(\beta+\varepsilon)} > Cq^{1/(\beta+\varepsilon)} > q(\delta).$$

We now inductively define an increasing sequence m_2, m_3, \dots as follows. Suppose that m_{i-1} has been defined such that

$$(54) \quad m_{i-1}^{\varepsilon/(\beta+\varepsilon)} > q(\delta).$$

Let

$$q_i := [m_{i-1}^{\varepsilon/(\beta+\varepsilon)}] + 1,$$

where $[x]$ denotes the integer part of a real number x . By (54), $q_i \geq q(\delta)$. Let $a_i := m_{i-1}/q_i$. Then if $q(\delta)$ is chosen large enough,

$$a_i \geq \frac{2}{3}m_{i-1}^{\beta/(\beta+\varepsilon)} \geq \frac{1}{2}q_i^{\beta/\varepsilon},$$

and

$$a_i \leq m_{i-1}^{\beta/(\beta+\varepsilon)} \leq q_i^{\beta/\varepsilon}.$$

Let r_i be an integer between $q_i^{(1-\beta-\varepsilon)/\varepsilon}$ and $2q_i^{(1-\beta-\varepsilon)/\varepsilon}$. Let $k_i = r_i q_i$ and $n_i = a_i k_i = a_i r_i q_i = r_i m_{i-1}$. If we carry out the argument of Section 7 with q_i, r_i, k_i, a_i, n_i in place of q, r, k, a, n , then, since $q_i \geq q(\delta)$, as before we arrive at the inequality

$$f(n_i) \geq (1 - \delta)r_i f(n_i/r_i) = (1 - \delta)r_i f(m_{i-1}).$$

Define $m_i := n_i$. Then the above inequality shows that

$$(55) \quad \frac{f(m_i)}{m_i} \geq (1 - \delta) \frac{f(m_{i-1})}{m_{i-1}}.$$

Note that since r_i is a positive integer and $m_i = r_i m_{i-1}$, therefore $m_i \geq m_{i-1}$. In particular, (54) is satisfied with m_i in place of m_{i-1} . This allows us to carry on the inductive construction such that (55) is satisfied for each i .

Now, the above construction shows that if the initial q was chosen large enough, then for each i ,

$$m_i = r_i m_{i-1} \geq q_i^{(1-\beta-\varepsilon)/\varepsilon} m_{i-1} \geq m_{i-1}^{1/(\beta+\varepsilon)}.$$

Therefore, for all $i \geq 2$,

$$m_i \geq m_1^{(\beta+\varepsilon)^{-(i-1)}}.$$

So, by (1), there exists a constant C_3 such that

$$\frac{f(m_i)}{m_i} \leq \frac{C}{\log m_i} \leq C_3(\beta + \varepsilon)^{i-1}.$$

However, (55) shows that there is $C_4 > 0$ such that

$$\frac{f(m_i)}{m_i} \geq C_4(1 - \delta)^{i-1}.$$

Since $1 - \delta > \beta + \varepsilon$, we get a contradiction for sufficiently large i .

9. PROOF OF $\chi \leq 2\xi - 1$ WHEN $\chi = 0$

As usual, we prove by contradiction. Assume that $\chi = 0$ and $2\xi - 1 < \chi$. Then $\xi < 1/2$. Choose ξ_1, ξ' and ξ'' such that $\xi < \xi_1 < \xi'' < \xi' < 1/2$. From this point on, however, the proof is quite different than the case $\chi > 0$. Recall that $t(P)$ is the sum of edge-weights of a path P in the environment $t = (t_e)_{e \in E(\mathbb{Z}^d)}$. This notation is used several times in this section. First, we need a simple lemma about the norm g .

Lemma 9.1. *Assume that the edge-weight distribution is continuous, and let L denote the infimum of its support. Then there exists $M > L$ such that for all $x \in \mathbb{R}^d \setminus \{0\}$, $g(x) \geq M|x|_1$, where $|x|_1$ is the ℓ_1 norm of x .*

Proof. Since g is a norm on \mathbb{R}^d ,

$$M := \inf_{x \neq 0} \frac{g(x)}{|x|_1} > 0,$$

and the infimum is attained. Choose $x \neq 0$ such that $g(x) = M|x|_1$. Define a new set of edge-weights s_e as $s_e := t_e - L$. Then s_e are non-negative and i.i.d. Let the function g^s be defined for these new edge-weights the same way g was defined for the old weights. Similarly, define h^s and T^s . Since any path P from a point y to a point z must have at least $|z - y|_1$ many edges, therefore $s(P) \leq t(P) - L|z - y|_1$. Thus,

$$T^s(y, z) \leq T(y, z) - L|z - y|_1.$$

In particular, $h^s(y) \leq h(y) - L|y|_1$ for any y . Considering a sequence y_n in \mathbb{Z}^d such that $y_n/n \rightarrow x$, we see that

$$\begin{aligned} g^s(x) &= \lim_{n \rightarrow \infty} \frac{h^s(y_n)}{n} \leq \lim_{n \rightarrow \infty} \frac{h(y_n) - L|y_n|_1}{n} \\ &= g(x) - L|x|_1 = (M - L)|x|_1. \end{aligned}$$

Since t_e has a continuous distribution, s_e has no mass at 0. Therefore, by a well-known result (see [17]), $g^s(x) > 0$. This shows that $M > L$. \square

Choose β , ε' and ε so small that $0 < \varepsilon' < \varepsilon < \beta < (\xi'' - \xi_1)/d$. Choose x_0 and H_0 as in Proposition 5.1. Let n be a positive integer, to be chosen arbitrarily large at the end of the proof. Again, as usual, C denotes any positive constant that does not depend on our choice of n .

Choose a point $z \in H_0$ such that $|z| \in [n^{\xi'}, 2n^{\xi'}]$. Let $v := nx_0/2 + z$. Then by Proposition 5.1 and the fact that $\xi' < 1/2$,

$$(56) \quad \begin{aligned} |g(v) - g(nx_0/2)| &= (n/2)|g(x_0 + 2z/n) - g(x_0)| \\ &\leq C|z|^2/n \leq Cn^{2\xi'-1} \leq C. \end{aligned}$$

Similarly,

$$(57) \quad |g(nx_0 - v) - g(nx_0/2)| \leq Cn^{2\xi'-1} \leq C.$$

Let w be the closest lattice point to v and let y be the closest lattice point to nx_0 . Then $|w - v|$ and $|y - nx_0|$ are bounded by \sqrt{d} . Therefore inequalities (56) and (57) imply that

$$(58) \quad |g(y) - (g(w) + g(y - w))| \leq C.$$

Figure 8 has an illustration of the relative locations of y and w , together with some other objects that will be defined below.

By Theorem 4.1 and the assumption that $\chi = 0$, $|h(y) - g(y)|$, $|h(w) - g(w)|$ and $|h(y - w) - g(y - w)|$ are all bounded by Cn^ε . Again by (A1) of Theorem 1.1 and the assumption that $\chi = 0$, the probabilities $\mathbb{P}(|T(0, w) - h(w)| > n^\varepsilon)$, $\mathbb{P}(|T(w, y) - h(y - w)| > n^\varepsilon)$ and $\mathbb{P}(|T(0, y) - h(y)| > n^\varepsilon)$ are all bounded by $e^{-Cn^{\varepsilon-\varepsilon'}}$. These observations, together with (58), imply that there are constants C_1 and C_2 , independent of our choice of n , such that

$$(59) \quad \mathbb{P}(|T(0, y) - (T(0, w) + T(w, y))| > C_1 n^\varepsilon) \leq e^{-C_2 n^{\varepsilon-\varepsilon'}}.$$

Let $T_o(0, y)$ be the minimum passage time from 0 to y among all paths that do not deviate by more than $n^{\xi''}$ from the straight line segment joining 0 and y . By assumption (A2) of Theorem 1.1,

$$\mathbb{P}(T_o(0, y) = T(0, y)) \geq 1 - e^{-Cn^{\xi''-\xi_1}}.$$

Combining this with (59), we see that if E_1 is the event

$$(60) \quad E_1 := \{|T_o(0, y) - (T(0, w) + T(w, y))| \leq C_1 n^\varepsilon\},$$

where C_1 is the constant from (59), then there is a constant C_3 such that

$$(61) \quad \mathbb{P}(E_1) \geq 1 - e^{-C_3 n^{\xi''-\xi_1}} - e^{-C_3 n^{\varepsilon-\varepsilon'}}.$$

Let V be the set of all lattice points within ℓ_1 distance n^β from w . Let ∂V denote the boundary of V in \mathbb{Z}^d , that is, all points in V that have at least one neighbor outside of V . Let w_1 be the first point in $G(0, w)$ that belongs to ∂V , when the points are arranged in a sequence from 0 to w . Let w_2 be the last point in $G(w, y)$ that belongs to ∂V , when the points are arranged

in a sequence from w to y . Let G_1 denote the portion of $G(0, w)$ connecting w_1 and w , and let G_2 denote the portion of $G(w, y)$ connecting w and w_2 . Let G_0 be the portion of $G(0, w)$ from 0 to w_1 and let G_3 be the portion of $G(w, y)$ from w_2 to y . Note that G_0 and G_3 lie entirely outside of V . Figure 8 provides a schematic diagram to illustrate the above definitions.

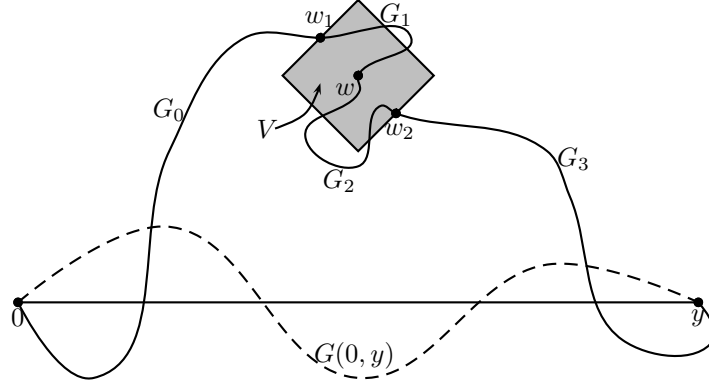


FIGURE 8. Schematic diagram for V, w, w_1, w_2 and G_0, G_1, G_2, G_3 .

Let L and M be as in Lemma 9.1. Choose L', M' such that $L < L' < M' < M$. Take any $u \in \partial V$. By Lemma 9.1, $g(u - w) \geq M|u - w|_1$. Therefore by Theorem 4.1,

$$h(u - w) \geq M|u - w|_1 - C|u - w|^\varepsilon \geq M|u - w|_1 - Cn^{\beta\varepsilon}.$$

Now, $|u - w|_1 \geq Cn^\beta$. Therefore by assumption (A1) of Theorem 1.1 and the above inequality,

$$\begin{aligned} \mathbb{P}(T(u, w) < M'|u - w|_1) \\ &\leq \mathbb{P}(|T(u, w) - h(u - w)| > (M - M')|u - w|_1 - Cn^{\beta\varepsilon}) \\ &\leq \mathbb{P}(|T(u, w) - h(u - w)| > Cn^\beta) \leq e^{-n^{\beta-\varepsilon'}/C}. \end{aligned}$$

Since there are at most n^C points in ∂V , the above bound shows that

$$\mathbb{P}(T(u, w) < M'|u - w|_1 \text{ for some } u \in \partial V) \leq n^C e^{-n^{\beta-\varepsilon'}/C}.$$

In particular, if E_2 and E_3 are the events

$$E_2 := \{t(G_1) \geq M'|w - w_1|_1\},$$

$$E_3 := \{t(G_2) \geq M'|w - w_2|_1\},$$

then there is a constant C_4 such that

$$(62) \quad \mathbb{P}(E_2 \cap E_3) \geq 1 - n^{C_4} e^{-n^{\beta-\varepsilon'}/C_4}.$$

Let $E(V)$ denote the set of edges between members of V . Let $(t'_e)_{e \in E(V)}$ be a collection of i.i.d. random variables, independent of the original edge-weights, but having the same distribution. For $e \notin E(V)$, let $t'_e := t_e$. Let E_4 be the event

$$E_4 := \{t'_e \leq L' \text{ for each } e \in E(V)\}.$$

If E_4 happens, then there is a path P_1 from w_1 to w and a path P_2 from w to w_2 such that $t'(P_1) \leq L'|w - w_1|_1$ and $t'(P_2) \leq L'|w - w_2|_1$. Let P be the concatenation of the paths G_0 , P_1 , P_2 and G_3 . Since $t'(G_0) = t(G_0)$ and $t'(G_3) = t(G_3)$, therefore under E_4 ,

$$t'(P) \leq t(G_0) + t(G_3) + L'|w - w_1|_1 + L'|w - w_2|_1.$$

On the other hand, under $E_2 \cap E_3$,

$$\begin{aligned} T(0, w) + T(w, y) &= t(G_0) + t(G_1) + t(G_2) + t(G_3) \\ &\geq t(G_0) + t(G_3) + M'|w - w_1|_1 + M'|w - w_2|_1. \end{aligned}$$

Consequently, if E_1, E_2, E_3, E_4 all happen simultaneously, then there is a (deterministic) positive constant C_5 such that

$$T_o(0, y) \geq t'(P) + C_5 n^\beta - C_1 n^\varepsilon,$$

where C_1 is the constant in the definition (60) of E_1 . Since $\beta < \xi'' < \xi'$ and $x_0 \notin H_0$, the edges within distance $n^{\xi''}$ of the line segment joining 0 and y have the same weights in the environment t' as in t . Since $\beta > \varepsilon$, this observation and the above display proves that $E_1 \cap E_2 \cap E_3 \cap E_4$ implies $D'(0, y) \geq n^{\xi''}$, where $D'(0, y)$ is the value of $D(0, y)$ in the new environment t' . (To put it differently, if $E_1 \cap E_2 \cap E_3 \cap E_4$ happens then there is a path P that has less t' -weight than the least t' -weight path within distance $n^{\xi''}$ of the straight line connecting 0 to y , and therefore $D'(0, y)$ must be greater than or equal to $n^{\xi''}$.)

Now note that the event E_4 is independent of E_1, E_2 and E_3 . Moreover, since $L' > L$, there is a constant C_6 such that $\mathbb{P}(E_4) \geq e^{-C_6 n^{\beta d}}$. Combining this with (61), (62) and the last observation from the previous paragraph, we get

$$\begin{aligned} \mathbb{P}(D'(0, y) \geq n^{\xi''}) &\geq \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) \\ &= \mathbb{P}(E_1 \cap E_2 \cap E_3) \mathbb{P}(E_4) \\ &\geq (1 - e^{-C_3 n^{\xi'' - \xi_1}} - e^{-C_3 n^{\varepsilon - \xi'}} - n^{C_4} e^{-n^{\beta - \varepsilon'}} / C_4) e^{-C_6 n^{\beta d}} \\ &\geq e^{-C_7 n^{\beta d}}. \end{aligned}$$

Now $D'(0, y)$ has the same distribution as $D(0, y)$. But by (A2) of Theorem 1.1, $\mathbb{P}(D(0, y) \geq n^{\xi''}) \leq e^{-C_8 n^{\xi'' - \xi_1}}$, and $\beta d < \xi'' - \xi_1$ by our choice of β . Together with the above display, this gives a contradiction, thereby proving that $\chi \leq 2\xi - 1$ when $\chi = 0$.

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